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GEOMETRY.

EUCLID'S ELEMENTS, BOOK I.

Definitions.

1. A **point** is that which has position, but not magnitude.

2. A **line** is length without breadth.

3. The **extremities of a line** are points.

4. A **straight line** is that which lies evenly between its extreme points.

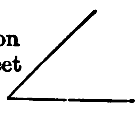
5. A **superficies** (or **surface**) is that which has only length and breadth.

6. The **extremities of a superficies** are lines.

7. A **plane superficies** is that in which any two points being taken, the straight line between them lies wholly in that superficies.

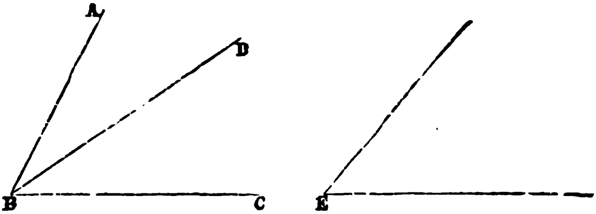
8. A **plane angle** is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A **plane rectilineal angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

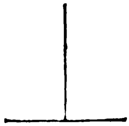


NOTE.—When several angles are at one point B, any one of them is expressed by three letters, of which the middle letter is B, and the first letter is on one of the straight lines which contain the angle, and the last letter on the other line.

Thus, the angle contained by the straight lines AB and BC is pressed either by ABC or CBA, and the angle contained by AB

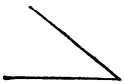


BD is expressed either by ABD or DBA. When there is only on angle at any given point, it may be expressed by the letter at tha point, as the angle E.



10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle**; and the straight line which stands on the other is called a **perpendicular** to it.

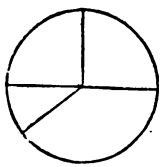
11. An **obtuse angle** is that which is greater than a right angle.



12. An **acute angle** is that which is less than a right angle.

13. A **term** or **boundary** is the extremity of anything.

14. A **figure** is that which is enclosed by one or more boundaries.



15. A **circle** is a plane figure contained by one line, which is called the **circumference**, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

16. And this point is called the **centre** of the circle, [and any straight line drawn from the centre to the circumference is called a **radius** of the circle].

17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

18. A **semicircle** is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. A **segment** of a circle is the figure contained by a straight line and the part of the circumference which it cuts off.

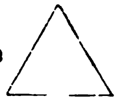
20. **Rectilineal figures** are those which are contained by straight lines.

21. **Trilateral figures, or triangles,** by three straight lines.

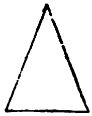
22. **Quadrilateral figures,** by four straight lines.

23. **Multilateral figures, or polygons,** by more than four straight lines.

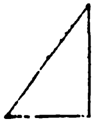
24. Of three-sided figures an **equilateral triangle** is that which has three equal sides.



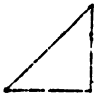
25. An **isosceles triangle** is that which has only two sides equal.



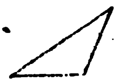
26. A **scalene triangle** is that which has three unequal sides.



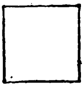
27. A **right-angled triangle** is that which has a right angle.



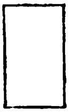
28. An **obtuse-angled triangle** is that which has an obtuse angle.



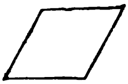
29. An **acute-angled triangle** is that which has three acute angles.



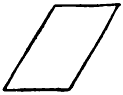
30. Of four-sided figures, a **square** is that which has all its sides equal, and all its angles right angles.



31. An **oblong** is that which has all its angles right angles, but not all its sides equal.



32. A **rhombus** is that which has all its sides equal, but its angles are not right angles.



33. A **rhomboid** is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

34. **Parallel straight lines** are such as are in the same plane, and which being produced ever so far both ways do not meet.

35. A **parallelogram** is a four-sided figure of which the opposite sides are parallel; and the **diagonal** is the straight line joining two of its opposite angles. All other four-sided figures are called **trapeziums.**

Postulates.

1. Let it be granted that a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. And that a circle may be described from any centre, at any distance from that centre.

Axioms.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same are equal to one another.

7. Things which are halves of the same are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot inclose a space.

11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced shall at length meet on that side on which are the angles which are less than two right angles.

Explanation of Terms and Abbreviations.

An **Axiom** is a truth admitted without demonstration.

A **Theorem** is a truth which is capable of being demonstrated from previously demonstrated or admitted truths.

A **Postulate** states a geometrical process, the power of effecting which is required to be admitted.

A **Problem** proposes to effect something by means of admitted processes, or by means of processes or constructions, the power of effecting which has been previously demonstrated.

A **Corollary** to a proposition is an inference which may be easily deduced from that proposition.

The sign $=$ is used to express *equality*.

$\angle$ means *angle*, and $\triangle$ signifies *triangle*.

The sign $>$ signifies "is greater than," and $<$ "is less than."

$+$ expresses addition; thus AB + BC is the line whose length is the *sum* of the lengths of AB and BC.

$-$ expresses subtraction; thus AB − BC is the excess of the length of the line AB above that of BC.

AB^2 means the square described upon the straight line AB.

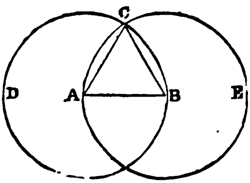
Proposition 1.—Problem.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line.

It is required to describe an equilateral triangle on AB.

From centres A and B, and radius = AB, describe circles.



CONSTRUCTION.—From the centre A, at the distance AB, describe the circle BCD (Post. 3).

From the centre B, at the distance BA, describe the circle ACE (Post. 3).

From the point C, in which the circles cut one another, draw the straight lines CA, CB to the points A and B (Post. 1).

Then ABC *shall be an equilateral triangle.*

AC=AB. PROOF.—Because the point A is the centre of the circle BCD, AC is equal to AB (Def. 15).

BC=AB. Because the point B is the centre of the circle ACE, BC is equal to BA (Def. 15).

AC and BC each=AB. Therefore AC and BC are each of them equal to AB.

AC=BC. But things which are equal to the same thing are equal to one another. Therefore AC is equal to BC (Ax. 1).

∴ AB=BC =CA. Therefore AB, BC, and CA are equal to one another.

Therefore the triangle ABC is equilateral, and it is described on the given straight line AB. *Which was to be done.*

Proposition 2.—Problem.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

It is required to draw from the point A a straight line equal to BC.

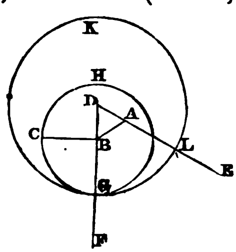
Construction.—From the point A to B draw the straight line AB (Post. 1). Draw AB.

Upon AB describe the equilateral triangle DAB (Book I., Prop. 1). Δ DAB equilateral.

Produce the straight lines DA, DB, to E and F (Post. 2).

From the centre B, at the distance BC, describe the circle CGH, meeting DF in G (Post. 3). B as centre.

From the centre D, at the distance DG, describe the circle GKL, meeting DE in L (Post. 3). D as centre.



Then AL *shall be equal to* BC.

Proof.—Because the point B is the centre of the circle CGH, BC is equal to BG (Def. 15). BC=BG.

Because the point D is the centre of the circle GKL, DL is equal to DG (Def. 15). DL=DG.

But DA, DB, parts of them, are equal (Construction). DA=DB.

Therefore the remainder AL is equal to the remainder BG (Ax. 3). ∴ AL = BG.

But it has been shown that BC is equal to BG.

Therefore AL and BC are each of them equal to BG. ∴ AL and BC each = BG.

But things which are equal to the same thing are equal to one another, therefore AL is equal to BC (Ax. 1).

Therefore from the given point A a straight line AL has been drawn equal to the given straight line BC. *Which was to be done.* ∴ AL = BC.

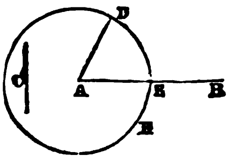
Proposition 3.—Problem.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB, the greater, a part equa to C, the less.

Make AD = C.



Construction.—From the point A draw the straight line AD equal to C (I. 2).

A as centre and radius AD.

From the centre A, at the distance AD, describe the circle DEF, cutting AB in E (Post. 3).

Then AE *shall be equal to* C.

Proof.—Because the point A is the centre of the circle

AE=AD. DEF, AE is equal to AD (Def. 15).

AD=C. But C is also equal to AD (Construction).

AE and C each=AD. Therefore AE and C are each of them equal to AD.

∴ AE=C. Therefore AE is equal to C (Ax. 1).

Therefore, from AB, the greater of two given straight lines, a part AE has been cut off, equal to C, the less. *Q. E. F.**

Proposition 4.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another: they shall have their bases, or third sides, equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, viz., those to which the equal sides are opposite. Or,

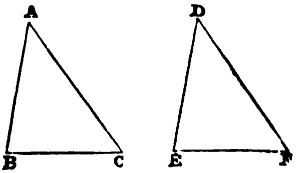
If two sides and the contained angle of one triangle be respectively equal to those of another, the triangles are equal in every respect.

Let ABC, DEF be two triangles which have

AB=DE. The two sides AB, AC, equal to the two sides DE, DF, each to each, viz., AB equal to DE, and AC equal to DF.

AC=DF.

∠ BAC = ∠ EDF.



And the angle BAC equal to the angle EDF :—then—

The base BC shall be equal to the base EF ;

The triangle ABC shall be equal to the triangle DEF ;

* Q. E. F. is an abbreviation for *quod erat faciendum*, that is "*which was to be done.*"

And the other angles to which the equal sides are opposite, shall be equal, each to each, viz., the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

PROOF.—For if the triangle ABC be applied to (*or placed upon*) the triangle DEF, Suppose Δ ABC put upon Δ DEF.

So that the point A may be on the point D, and the straight line AB on the straight line DE,

The point B shall coincide with the point E, because AB is equal to DE (Hypothesis).

And AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF (Hyp.).

Therefore also the point C shall coincide with the point F, because the straight line AC is equal to DF (Hyp.).

But the point B was proved to coincide with the point E.

Therefore the base BC shall coincide with the base EF.

Because the point B coinciding with E, and C with F, if the base BC do not coincide with the base EF, two straight lines would enclose a space, which is impossible (Ax. 10).

Therefore the base BC coincides with the base EF, and is therefore equal to it (Ax. 8). BC=EF.

Therefore the whole triangle ABC coincides with the whole triangle DEF, and is equal to it (Ax. 8). ∴ Δ ABC = Δ DEF.

And the other angles of the one coincide with the remaining angles of the other, and are equal to them, viz., the angle ABC to DEF, and the angle ACB to DFE. ∠ ABC = ∠ DEF. ∠ ACB = ∠ DFE.

Therefore, if two triangles have, &c. (see Enunciation). *Which was to be shown.*

Proposition 5.—Theorem.

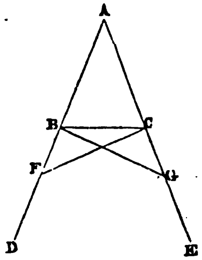
The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall also be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to the side AC. AB = AC.

Let the straight lines AB, AC (*the equal sides of the triangle*), be produced to D and E.

The angle ABC shall be equal to the angle ACB (*angles at the base*),

And the angle CBD shall be equal to the angle BCE (*angles upon the other side of the base*).



CONSTRUCTION.—In BD take any point F.

AG = AF. From AE, the greater, cut off AG, equal to AF, the less (I. 3).

Join FC, GB.

PROOF.—Because AF is equal to AG (Construction), and AB is equal to AC (Hyp.),

FA, AC respectively = GA, AB. Therefore the two sides FA, AC are equal to the two sides GA, AB, each to each;

And they contain the angle FAG, common to the two triangles AFC, AGB.

∴ FC = GB and △ AFC = △ AGB. Therefore the base FC is equal to the base GB (I. 4);

And the triangle AFC to the triangle AGB (I. 4);

∠ ACF = ∠ ABG. ∠ AFC = ∠ AGB. And the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite, viz., the angle ACF to the angle ABG, and the angle AFC to the angle AGB (I. 4).

And because the whole AF is equal to the whole AG, of which the parts AB, AC, are equal (Hyp.),

BF = CG. The remainder BF is equal to the remainder CG (Ax. 3).

And FC was proved to be equal to GB;

Therefore the two sides BF, FC are equal to the two sides CG, GB, each to each.

And the angle BFC was proved equal to the angle CGB;

Therefore the triangles BFC, CGB are equal; and their other angles are equal, each to each, to which the equal sides are opposite (I. 4).

∴ ∠ FBC = ∠ GCB. ∠ BCF = ∠ CBG. Therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG.

And since it has been demonstrated that the whole angle ABG is equal to the whole angle ACF, and that the parts of these, the angles CBG, BCF, are also equal,

∴ ∠ ABC = ∠ ACB. Therefore the remaining angle ABC is equal to the remaining angle ACB (Ax. 3),

...h are the angles at the base of the triangle ABC.

And it has been proved that the angle FBC is equal to the angle GCB (Dem. 11),

Which are the angles upon the other side of the base,

Therefore the angles at the base, &c. (see Enunciation). *Which was to be shown.*

Corollary.—Hence every equilateral triangle is also equiangular.

Proposition 6.—Theorem.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB.

The side AB shall be equal to the side AC.

For if AB be not equal to AC, one of them is greater than the other. Let AB be the greater. Suppose AB > AC.

Construction.—From AB, the greater, cut off a part DB, equal to AC, the less (I. 3). Make DB = AC.

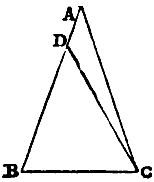
Join DC.

Proof.—Because in the triangles DBC, ACB, DB is equal to AC, and BC is common to both,

Therefore the two sides DB, BC are equal to the two sides AC, CB, each to each;

And the angle DBC is equal to the angle ACB (Hyp.)

Therefore the base DC is equal to the base AB (I. 4).



And the triangle DBC is equal to the triangle ACB (I. 4), the less to the greater, which is absurd. Δ DBC = Δ ACB.

Therefore AB is not unequal to AC, that is, it is equal to it.

Wherefore, if two angles, &c. *Q. E. D.* *

Corollary.—Hence every equiangular triangle is also equilateral.

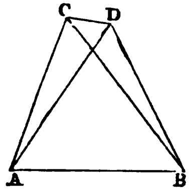
* Q. E. D. is an abbreviation for *quod erat demonstrandum*, that is, "*which was to be shown or proved.*"

Proposition 7.—Theorem.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides, which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

Let the triangles ACB, ADB, upon the same base AB, and on the same side of it, have, if possible,

Suppose CA = DA.



Their sides CA, DA, terminated in the extremity A of the base, equal to one another;

CB = DB.

And their sides CB, DB, terminated in the extremity B of the base, likewise equal to one another.

Case I.—Let the vertex of each triangle be without the other triangle.

Construction.—Join CD.

Proof.—Because AC is equal to AD (Hyp.),

The triangle ADC is an isosceles triangle, and the angle

∠ ACD = ∠ ADC.

ACD is therefore equal to the angle ADC (I. 5).

But the angle ACD is greater than the angle BCD (Ax. 9).

Therefore the angle ADC is also greater than BCD.

BDC > BCD.

Much more then is the angle BDC greater than BCD.

Again, because BC is equal to BD (Hyp.),

The triangle BCD is an isosceles triangle, and the angle

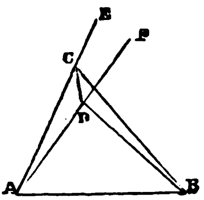
∠ BDC = ∠ BCD.

BDC is equal to the angle BCD (I. 5).

But the angle BDC has been shown to be greater than the angle BCD (Dem. 5).

∠ BDC = and > ∠ BCD.

Therefore the angle BDC is both equal to, and greater than the same angle BCD, which is impossible.



Case II.—Let the vertex of one of the triangles fall within the other.

Construction.—Produce AC, AD to E and F, and join CD.

Proof.—Because AC is equal to AD (Hyp.),

Again ∠ ECD = ∠ FDC.

The triangle ADC is an isosceles triangle, and the angles ECD, FDC, upon the other side of its base CD, are equal to one another (I. 5).

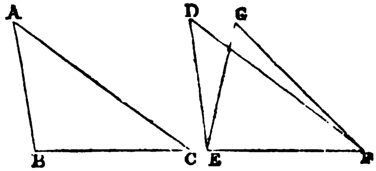
But the angle ECD is greater than the angle BCD (Ax. 9).
Therefore the angle FDC is likewise greater than BCD.
Much more then is the angle BDC greater than BCD. $\angle$ BDC > $\angle$ BCD.
Again, because BC is equal to BD (Hyp.),
The triangle BDC is an isosceles triangle, and the angle BDC is equal to the angle BCD (I. 5). $\angle$ BDC = $\angle$ BCD.
But the angle BDC has been shown to be greater than the angle BCD.
Therefore the angle BDC is both equal to, and greater than the same angle BCD, which is impossible. $\therefore$ $\angle$ BDC = and > $\angle$ BCD.
Therefore, upon the same base, &c. *Q. E. D.*

Proposition 8.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other. Or,

If two triangles have three sides of the one respectively equal to the three sides of the other, they are equal in every respect, those angles being equal which are opposite to the equal sides.

Let ABC, DEF be two triangles which have
The two sides AB, AC equal to the two sides DE, DF, each to each, viz., AB to DE, and AC to DF, Given AB = DE, AC = DF,
And the base BC equal to the base EF. and BC = EF.
The angle BAC shall be equal to the angle EDF.
PROOF.—For if the triangle ABC be applied to the triangle DEF,
So that the point B may be on E, and the straight line BC on EF,



The point C shall coincide with the point F, because BC is equal to EF (Hyp.). Make BC coincide with EF.
Therefore, BC coinciding with EF, BA and AC shall coincide with ED and DF.
For if the base BC coincides with the base EF,

But the sides BA, AC, do not coincide with the sides ED, DF, but have a different situation, as EG, GF,

Then upon the same base, and on the same side of it, there will be two triangles, which have their sides terminated in one extremity of the base equal to one another, and likewise their sides, which are terminated in the other extremity. But this is impossible (I. 7).

∴ BA, AC respectively coincide with ED, DF.

Therefore, if the base BC coincides with the base EF, the sides BA, AC must coincide with the sides ED, DF.

Therefore the angle BAC coincides with the angle EDF, and is equal to it (Ax. 8).

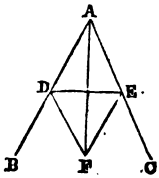
Also the triangle ABC coincides with the triangle DEF and is therefore equal to it in every respect (Ax. 8).

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 9.—Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.

Let BAC be the given rectilineal angle.



It is required to bisect it.

Construction.—Take any point D in AB.

Make AE = AD.

From AC cut off AE equal to AD (I. 3).

Join DE.

△ DEF equilateral.

Upon DE, on the side remote from A, describe an equilateral triangle DEF (I. 1).

Join AF.

Then the straight line AF *shall bisect the angle* BAC.

Proof.—Because AD is equal to AE (Const.), and AF is common to the two triangles DAF, EAF;

The two sides DA, AF are equal to the two sides EA, AF, each to each;

And the base DF is equal to the base EF (Const.);

∴ ∠DAF = ∠EAF.

Therefore the angle DAF is equal to the angle EAF (I. 8).

Therefore the given rectilineal angle BAC is bisected by the straight line AF. *Q. E. F.*

Proposition 10.—Problem.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line.

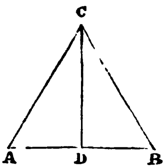
It is required to divide it into two equal parts.

Construction.—Upon AB describe the equilateral triangle ABC (I. 1).

Make Δ ABC equilateral and ∠ ACD = ∠ BCD.

Bisect the angle ACB by the straight line CD (I. 9).

Then AB *shall be cut into two equal parts in the point* D.



Proof.—Because AC is equal to CB (Const.), and CD common to the two triangles ACD, BCD;

The two sides AC, CD are equal to the two sides BC, CD, each to each;

And the angle ACD is equal to the angle BCD (Const.);

Therefore the base AD is equal to the base DB (I. 4). ∴ AD = DB.

Therefore the straight line AB is divided into two equal parts in the point D. *Q. E. F.*

Proposition 11.—Problem.

To draw a straight line at right angles to a given straight line from a given point in the same.

Let AB be the given straight line, and C a given point in it.

It is required to draw a straight line from the point C at right angles to AB.

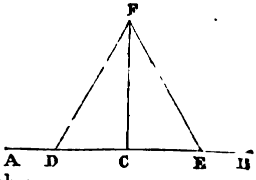
Construction.—Take any point D in AC.

Make CE equal to CD (I. 3).

Upon DE describe the equilateral triangle DFE (I. 1). Make CE = CD and Δ DEF equilateral.

Join FC.

Then FC *shall be at right angles to* AB.



Proof.—Because DC is equal to CE (Const.), and FC common to the two triangles DCF, ECF;

The two sides DC, CF, are equal to the two sides EC, CF, each to each;

And the base DF is equal to the base EF (Const.);

Therefore the angle DCF is equal to the angle ECF (I. 8); ∠ DCF = ∠ ECF.

And they are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle (Def. 10);

∴ ∠ DCF, ∠ ECF are right angles.

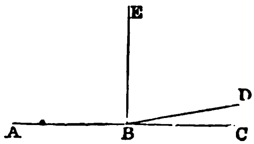
Therefore each of the angles DCF, ECF is a right angle.

Therefore from the given point C in the given straight line AB, a straight line FC has been drawn at right angles to AB. *Q. E. F.*

Corollary.—By help of this problem, it may be demonstrated that

Two straight lines cannot have a common segment.

If it be possible, let the two straight lines ABC, ABD, have the segment AB common to both of them.



Make ∠ ABE a right ∠

Construction.—From the point B, draw BE at right angles to AB (I. 11).

∠ CBE = ∠ EBA.

Proof. — Because ABC is a straight line, the angle CBE is equal to the angle EBA (Def. 10).

and ∠ DBE = ∠ EBA.

Also, because ABD is a straight line, the angle DBE is equal to the angle EBA (Def. 10).

∴ ∠ DBE = ∠ CBE.

Therefore the angle DBE is equal to the angle CBE. The less to the greater; which is impossible.

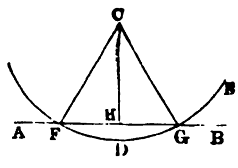
Therefore two straight lines cannot have a common segment.

Proposition 12.—Problem.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it.

It is required to draw from the point C, a straight line perpendicular to AB.



Construction.—Take any point D upon the other side of AB.

CD as radius.

From the centre C, at the distance CD, describe the circle EGF, meeting AB in F and G (Post. 3).

Bisect FG in H.

Bisect FG in H (I. 10).

Join CF, CH, CG.

Then CH *shall be perpendicular to* AB.

Proof.—Because FH is equal to HG Const.), and HC common to the two triangles FHC, GHC.

The two sides FH, HC are equal to the two sides GH, HC, each to each;

And the base CF is equal to the base CG (Def. 15);

Therefore the angle CHF is equal to the angle CHG (I. 8), and they are adjacent angles.

∴ adjacent angles CHF, CHG are equal.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it (Def. 10).

Therefore, from the given point C, a perpendicular has been drawn to the given straight line AB. *Q. E. F.*

Proposition 13.—Theorem.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD.

These angles shall either be two right angles, or shall together be equal to two right angles.

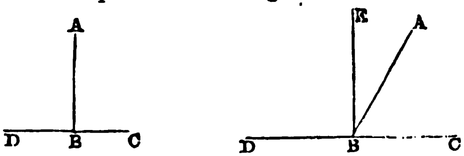
Proof.—If the angle CBA be equal to the angle ABD, each of them is a right angle (Def. 10).

But if the angle CBA be not equal to the angle ABD, from the point B, draw BE at right angles to CD (I. 11).

Therefore the angles CBE, EBD, are two right angles.

Now the angle CBE is equal to the two angles CBA, ABE; to each of these equals add the angle EBD.

Make ∠ CBE = ∠ EBD = a right ∠.



Therefore the angles CBE, EBD, are equal to the three angles CBA, ABE, EBD (Ax. 2).

Again, the angle DBA is equal to the two angles DBE, EBA; to each of these equals add the angle ABC.

Therefore the angles DBA, ABC, are equal to the three angles DBE, EBA, ABC (Ax. 2).

∴ ∠ CBE + ∠ EBD = ∠ CBA + ∠ ABE + ∠ EBD, also ∠ DBA + ∠ ABC = ∠ DBE + ∠ EBA + ∠ ABC.

But the angles CBE, EBD have been shown to be equal to the same three angles;

And things which are equal to the same thing are equal to one another;

∴ ∠ CBE + ∠ EBD = ∠ DBA + ∠ ABC.

Therefore the angles CBE, EBD, are equal to the angles DBA, ABC (Ax. 1).

But the angles CBE, EBD are two right angles.

Therefore the angles DBA, ABC, are together equal to two right angles (Ax. 1).

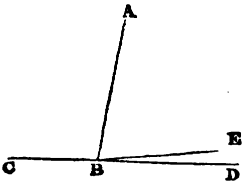
Therefore, the angles which one straight line, &c. *Q. E. D.*

Proposition 14.—Theorem.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

Given ∠ ABC + ∠ ABD = two right angles.

At the point B in the straight line AB, let the two straight lines BC, BD, upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.



BD shall be in the same straight line with BC.

If possible, let CBE be a straight line.

For if BD be not in the same straight line with BC, let BE be in the same straight line with it.

Proof.—Because CBE is a straight line, and AB meets it in B.

Therefore the adjacent angles ABC, ABE are together equal to two right angles (I. 13).

But the angles ABC, ABD, are also together equal to two right angles (Hyp.);

Therefore the angles ABC, ABE, are equal to the angles ABC, ABD (Ax. 1).

Take away the common angle ABC.

∴ ∠ ABE = ∠ ABD.

The remaining angle ABE is equal to the remaining angle ABD (Ax. 3), the less to the greater, which is impossible;

Therefore BE is not in the same straight line with BC.

And, in like manner, it may be demonstrated that no other can be in the same straight line with it but BD.

Therefore BD is in the same straight line with BC.

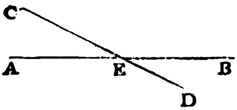
Therefore, if at a point, &c. *Q. E. D.*

Proposition 15.—Theorem.*

If two straight lines cut one another, the vertical, or opposite angles shall be equal.

Let the two straight lines AB, CD cut one another in the point E.

The angle AEC shall be equal to angle DEB, and the angle CEB to the angle AED.



PROOF.—Because the straight line AE makes with CD, the angles CEA, AED, these angles are together equal to two right angles (I. 13).

∠ CEA + ∠ AED = 2 right angles.

Again, because the straight line DE makes with AB the angles AED, DEB, these also are together equal to two right angles (I. 13).

∠ AED + ∠ DEB = 2 right angles.

But the angles CEA, AED have been shown to be together equal to two right angles,

Therefore the angles CEA, AED are equal to the angles AED, DEB (Ax. 1).

Take away the common angle AED.

The remaining angle CEA is equal to the remaining angle DEB (Ax. 3).

∴ ∠ CEA = ∠ DEB.

In the same manner it can be shown that the angles CEB, AED are equal.

Therefore, if two straight lines, &c. *Q. E. D.*

COROLLARY 1.—From this it is manifest that if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.

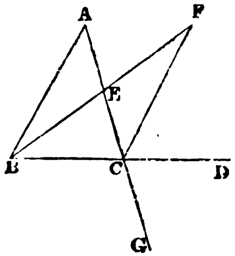
COROLLARY 2.—And, consequently, that all the angles made by any number of lines meeting in one point are together equal to four right angles, provided that no one of the angles be included in any other angle.

Proposition 16.—Theorem.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D.

The exterior angle ACD shall be greater than either of the interior opposite angles CBA, BAC.



Make AE = EC. and EF = BE.

CONSTRUCTION.—Bisect AC in E (I. 10).

Join BE, and produce it to F, making EF equal to BE (I. 3), and join FC

PROOF.—Because AE is equal to EC, and BE equal to EF (Const.),

AE, EB are equal to CE, EF, each to each ;

And the angle AEB is equal to the angle CEF, because they are opposite vertical angles (I. 15).

Therefore the base AB is equal to the base CF (I. 4);

And the triangle AEB to the triangle CEF (I. 4);

And the remaining angles to the remaining angles, each to each, to which the equal sides are opposite.

$\therefore \angle$BAE $= \angle$ ECF. Therefore the angle BAE is equal to the angle ECF (I. 4).

But the angle ECD is greater than the angle ECF (Ax. 9);

Therefore the angle ACD is greater than the angle BAE.

$\therefore \angle$ACD $> \angle$ BAE. In the same manner, if BC be bisected, and the side AC be produced to G, it may be proved that the angle BCG (or its equal ACD), is greater than the angle ABC.

Therefore, if one side, &c. *Q. E. D.*

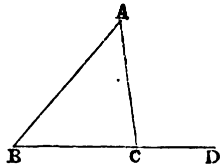
Proposition 17.—Theorem.

Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle.

Any two of its angles together shall be less than two right angles

CONSTRUCTION.—Produce BC to D.



PROOF.—Because ACD is the exterior angle of the triangle ABC, it is greater than the interior and opposite angle ABC (I. 16). ∠ ACD > ∠ ABC.

To each of these add the angle ACB. Add to each ∠ ACB.

Therefore the angles ACD, ACB are greater than the angles ABC, ACB (Ax. 4). ∠ ABC + ∠ ACB < 2 right angles.

But the angles ACD, ACB are together equal to two right angles (I. 13);

Therefore the angles ABC, ACB are together less than two right angles.

In like manner, it may be proved that the angles BAC, ACB, as also the angles CAB, ABC are together less than two right angles.

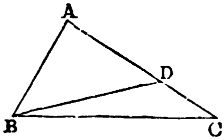
Therefore, any two angles, &c. *Q. E. D.*

Proposition 18.—Theorem.

The greater side of every triangle is opposite the greater angle.

Let ABC be a triangle, of which the side AC is greater than the side AB. AC > AB.

The angle ABC shall be greater than the angle BCA.



CONSTRUCTION.—Because AC is greater than AB, make AD equal to AB (I. 3), and join BD. Make AD = AB.

PROOF.—Because ADB is the exterior angle of the triangle BDC, it is greater than the interior and opposite angle BCD (I. 16). ∠ ADB > ∠ BCD, and

But the angle ADB is equal to the angle ABD; the triangle BAD being isosceles (I. 5), ∠ ADB = ∠ ABD and

Therefore the angle ABD is greater than the angle BCD (*or* ACB). ∴ ∠ ABD > ∠ BCD.

Much more then is the angle ABC greater than the angle ACB.

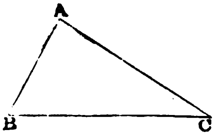
Therefore, the greater side, &c. *Q. E. D.*

Proposition 19.—Theorem.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Given ∠ ABC > ∠ BCA. Let ABC be a triangle, of which the angle ABC is greater than the angle BCA;

The side AC shall be greater than the side AB.



PROOF.—If AC be not greater than AB, it must either be equal to or less than AB.

It is not equal, for then the angle ABC would be equal to the angle BCA (I. 5); but it is not (Hyp.);

AC not = AB. Therefore AC is not equal to AB.

Neither is AC less than AB, for then the angle ABC would be less than the angle BCA (I. 18); but it is not (Hyp.);

AC not < AB. Therefore AC is not less than AB.

And it has been proved that AC is not equal to AB;

Therefore AC is greater than AB.

Therefore, the greater angle, &c. *Q. E. D.*

Proposition 20.—Theorem.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle;

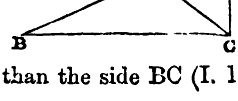
Any two sides of it are together greater than the third side.

Make AD = AC. CONSTRUCTION.—Produce BA to the point D, making AD equal to AC (I. 3), and join DC.

PROOF.—Because DA is equal to AC, the angle ADC is equal to the angle ACD (I. 5).

But the angle BCD is greater than the angle ACD (Ax. 9);

∠ BCD > ∠ BDC. Therefore the angle BCD is greater than the angle ADC (*or* BDC).



And because the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater angle is subtended by the greater side;

∴ DB > BC. Therefore the side DB is greater than the side BC (I. 19).

But BD is equal to BA and AC;
Therefore BA, AC are greater than BC. ∴ BA + AC > BC.
In the same manner it may be proved that AB, BC are greater than AC; and BC, CA greater than AB.
Therefore any two sides, &c. *Q. E. D.*

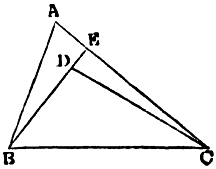
Proposition 21.—Theorem.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to the point D within the triangle;

BD, DC shall be less than the sides BA, AC;

But BD, DC shall contain an angle BDC greater than the angle BAC.



Construction.—Produce BD to E.

Proof.—1. Because two sides of a triangle are greater than the third side (I. 20), the two sides BA, AE, of the triangle BAE are greater than BE.

To each of these add EC.

Therefore the sides BA, AC, are greater than BE, EC (Ax. 4). BA + AC > BE + EC.

Again, because the two sides CE, ED, of the triangle CED are greater than CD (I. 20),

To each of these add DB.

Therefore CE, EB are greater than CD, DB (Ax. 4).

But it has been shown that BA, AC are greater than BE, EC; ∴ EC + EB > CD + DB.

Much more then are BA, AC greater than BD, DC.

Proof. — 2. Again, because the exterior angle of a triangle is greater than the interior and opposite angle (I. 16), therefore BDC, the exterior angle of the triangle CDE, is greater than CED or CEB. Again ∠ BDC > ∠ CEB,

For the same reason, CEB, the exterior angle of the triangle ABE, is greater than the angle BAE or BAC. and ∠ CEB > ∠ BAE.

And it has been shown that the angle BDC is greater than CEB;

∴ ∠ BDC > ∠ BAC. Much more then is the angle BDC greater than the angle BAC.

Therefore, if from the ends, &c. *Q. E. D.*

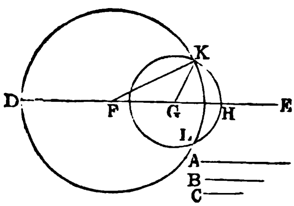
Proposition 22.—Problem.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these lines must be greater than the third (I. 20).

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third—namely, A and B greater than C, A and C greater than B, and B and C greater than A;

It is required to make a triangle of which the sides shall be equal to A, B, and C, each to each.

CONSTRUCTION.—Take a straight line DE terminated at the point D, but unlimited towards E.



DF, FG, GH respectively = A, B, C. Make DF equal to A, FG equal to B, and GH equal to C (I. 3).

FD as radius. From the centre F, at the distance FD, describe the circle DKL (Post. 3).

and GH as radius. From the centre G, at the distance GH, describe the circle HLK (Post. 3).

Join KF, KG.

Then the triangle KFG *shall have its sides equal to the three straight lines* A, B, C.

PROOF.—Because the point F is the centre of the circle DKL, FD is equal to FK (Def. 15).

But FD is equal to A (Const.);

FK = A. Therefore FK is equal to A (Ax. 1).

Again, because the point G is the centre of the circle HLK, GH is equal to GK (Def. 15).

But GH is equal to C (Const.);

GK = C. Therefore GK is equal to C (Ax. 1),

And FG is equal to B (Const.); FG = B

Therefore the three straight lines KF, FG, GK are equal to the three A, B, C, each to each.

Therefore the triangle KFG has its three sides KF, FG, GK equal to the three given straight lines A, B, C. *Q. E. F.*

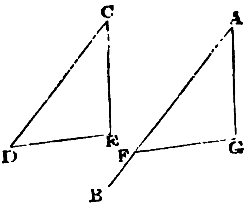
Proposition 23.—Problem.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle.

It is required to take an angle at the point A, in the straight line AB, equal to the rectilineal angle DCE.

CONSTRUCTION. — In CD, CE, take any points D, E, and join DE.



On AB construct a triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC — namely, AF equal to CD, FG to DE, and AG to EC (I. 22);

Make △ AFG so that AF = CD FG = DE AG = CE.

Then the angle FAG *shall be equal to the angle* DCE.

PROOF.—Because DC, CE are equal to FA, AG, each to each, and the base DE equal to the base FG (Const.),

The angle DCE is equal to the angle FAG (I. 8).

Then ∠ DCE = ∠ FAG

Therefore, at the given point A, in the given straight line AB, the angle FAG has been made equal to the given rectilineal angle DCE. *Q. E. F.*

Proposition 24.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

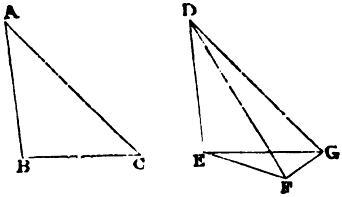
Let ABC, DEF, be two triangles which have

The two sides AB, AC equal to the two DE, DF, each to each—namely, AB to DE, and AC to DF,

But the angle BAC greater than the angle EDF;

The base BC shall be greater than the base EF.

Suppose DF > DE. CONSTRUCTION.—Let the side DF of the triangle DEF be greater than its side DE.



Make ∠ EDG = ∠ BAC. Then at the point D, in the straight line ED, make the angle EDG equal to the angle BAC (I. 23).

Make DG = AC, and = DF. Make DG equal to AC or DF (I. 3).

Join EG, GF.

PROOF.—Because AB is equal to DE (Hyp.), and AC to DG (Const.), the two sides BA, AC are equal to the two ED, DG, each to each;

And the angle BAC is equal to the angle EDG (Const.);

∴BC=EG. Therefore the base BC is equal to the base EG (I. 4).

And because DG is equal to DF (Const.), the angle DFG is equal to the angle DGF (I. 5).

But the angle DGF is greater than the angle EGF (Ax. 9);

and ∠ EFG > ∠ EGF. Therefore the angle DFG is greater than the angle EGF;

Much more then is the angle EFG greater than the angle EGF.

And because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater angle is subtended by the greater side,

∴EG>EF. Therefore the side EG is greater than the side EF (I. 19).

But EG was proved equal to BC;

Therefore BC is greater than EF.

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 25.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

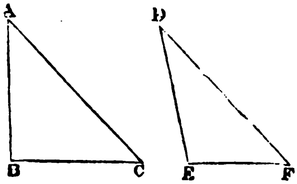
Let ABC, DEF, be two triangles, which have

The two sides AB, AC equal to the two sides DE, DF, each to each—namely, AB to DE, and AC to DF,

But the base BC greater than the base EF;

The angle BAC shall be greater than the angle EDF.

PROOF.—For if the angle BAC be not greater than the angle EDF, it must either be equal to it or less.



But the angle BAC is not equal to the angle EDF, for then the base BC would be equal to the base EF (I. 4), but it is not (Hyp.);

Therefore the angle BAC is not equal to the angle EDF; $\angle$ BAC not $= \angle$ EDF.

Neither is the angle BAC less than the angle EDF, for then the base BC would be less than the base EF (I. 24), but it is not (Hyp.),

Therefore the angle BAC is not less than the angle EDF. $\angle$ BAC not $< \angle$ EDF.

And it has been proved that the angle BAC is not equal to the angle EDF;

Therefore the angle BAC is greater than the angle EDF.

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 26.—Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side—namely, either the side adjacent to the equal angles in each, or the side opposite to them; then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other. Or,

If two angles and a side in one triangle be respectively equal to two angles and a corresponding side in another triangle, the triangles shall be equal in every respect.

Let ABC, DEF be two triangles, which have

The angles ABC, BCA equal to the angles DEF, EFD, each to each—namely, ABC to DEF, and BCA to EFD;

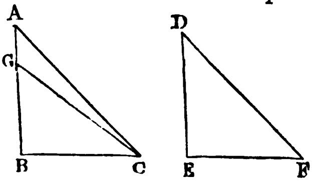
Also one side equal to one side.

CASE 1.—First, let the sides adjacent to the equal angles in each be equal—namely, BC to EF; Given BC = EF.

Then shall the side AB be equal to DE, the side AC to DF, and the angle BAC to the angle EDF.

Suppose $AB > DE$.

For if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater of the two.



Make $BG = DE$.

CONSTRUCTION.—Make BG equal to DE (I. 3), and join GC.

PROOF.—Because BG is equal to DE (Const.), and BC is equal to EF (Hyp.), the two sides GB, BC are equal to the two sides DE, EF, each to each.

And the angle GBC is equal to the angle DEF (Hyp.);
Therefore the base GC is equal to the base DF (I. 4),
And the triangle GBC to the triangle DEF (I. 4),
And the other angles to the other angles, each to each, to which the equal sides are opposite;

$\therefore \angle GCB = \angle DFE$.

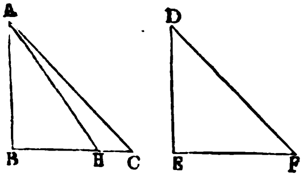
Therefore the angle GCB is equal to the angle DFE (I. 4).
But the angle DFE is equal to the angle BCA (Hyp.);
Therefore the angle GCB is equal to the angle BCA (Ax. 1), the less to the greater, which is impossible;

AB not unequal to DE.

Therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF (Hyp.);
Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each,
And the angle ABC is equal to the angle DEF (Hyp.);
Therefore the base AC is equal to the base DF (I. 4),
And the third angle BAC to the third angle EDF (I. 4).

CASE 2.—Next, let the sides which are opposite to the equal angles in each triangle be equal to one another—namely, AB equal to DE.

Likewise in this case the other sides shall be equal, AC to DF, and BC to EF; and also the angle BAC to the angle EDF.



Suppose $BC > EF$

For if BC be not equal to EF, one of them must be greater than the other. Let BC be the greater of the two.

Make $BH = EF$.

CONSTRUCTION.—Make BH equal to EF (I. 3), and join AH.

PROOF.—Because BH is equal to EF (Const.), and AB is equal to DE (Hyp.), the two sides AB, BH are equal to the two sides DE, EF, each to each, BH = EF, AB = DE.

And the angle ABH is equal to the angle DEF (Hyp.); ∠ ABH = ∠ DEF.

Therefore the base AH is equal to the base DF (I. 4),

And the triangle ABH to the triangle DEF (I. 4),

And the other angles to the other angles, each to each, to which the sides are opposite;

Therefore the angle BHA is equal to the angle EFD (I. 4). ∴ ∠ BHA

But the angle EFD is equal to the angle BCA (Hyp.); = ∠ EFD

Therefore the angle BHA is also equal to the angle BCA (Ax. 1); = ∠ BCA.

That is, the exterior angle BHA of the triangle AHC, is equal to its interior and opposite angle BCA, which is impossible (I. 16);

Therefore BC is not unequal to EF—that is, it is equal to it; and AB is equal to DE (Hyp.); BC not unequal to EF.

Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each,

And the angle ABC is equal to the angle DEF (Hyp.);

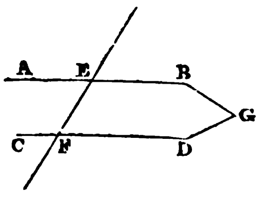
Therefore the base AC is equal to the base DF (I. 4),

And the third angle BAC is equal to the third angle EDF (I. 4).

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 27.—Theorem.

If a straight line falling upon two other straight lines make the alternate angles equal to one another, these two straight lines shall be parallel.



Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another. Given ∠ AEF = ∠ EFD.

AB shall be parallel to CD.

For if AB and CD be not parallel, they will meet if produced, either towards B, D, or towards A, C. Let them be produced, and meet towards B, D, in the point G.

∠ AEF > ∠ EFG, and also = ∠ EFG.

PROOF.—Then GEF is a triangle, and its exterior angle AEF is greater than the interior and opposite angle EFG (I. 16).

But the angle AEF is also equal to EFG (Hyp.), which is impossible;

Therefore AB and CD, being produced, do not meet towards B, D.

In like manner it may be shown that they do not meet towards A, C.

But those straight lines in the same plane which being produced ever so far both ways do not meet are parallel (Def. 34);

Therefore AB is parallel to CB.

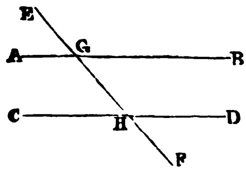
Therefore, if a straight line, &c. *Q. E. D.*

Proposition 28.—Theorem.

If a straight line falling upon two other straight lines make the exterior angle equal to the interior and opposite upon the same side of the line, or make the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make—

The exterior angle EGB equal to the interior and opposite angle GHD, upon the same side;



Or make the interior angles on the same side, the angles BGH, GHD, together equal to two right angles;

AB shall be parallel to CD.

PROOF 1.—Because the angle EGB is equal to the angle GHD (Hyp.),

And the angle EGB is equal to the angle AGH (I. 15);

∠ AGH = ∠ GHD.

Therefore the angle AGH is equal to the angle GHD (Ax. 1), and these angles are alternate;

Therefore AB is parallel to CD (I. 27).

PROOF 2.—Again, because the angles BGH, GHD are equal to two right angles (Hyp.),

And the angles BGH, AGH are also equal to two right angles (I. 13).

Therefore the angles BGH, AGH are equal to the angles BGH, GHD (Ax. 1). ∴ ∠ BGH + ∠ AGH = ∠ BGH + ∠ GHD.

Take away the common angle BGH.

Therefore the remaining angle AGH is equal to the remaining angle GHD (Ax. 3), and they are alternate angles. ∴ ∠ AGH = ∠ GHD.

Therefore AB is parallel to CD (I. 27).

Therefore, if a straight line, &c. *Q. E. D.*

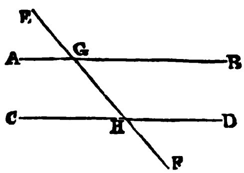
Proposition 29.—Theorem.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite upon the same side; and also the two interior angles upon the same side together equal to two right angles.

Let the straight line EF fall upon the parallel straight lines AB, CD;

The alternate angles AGH, GHD shall be equal to one another.

The exterior angle EGB shall be equal to GHD, the interior and opposite angle upon the same side;



And the two interior angles on the same side BGH, GHD shall be together equal to two right angles.

For if AGH be not equal to GHD, one of them must be greater than the other. Let AGH be the greater. ∠ AGH > GHD. (suppose.)

PROOF.—Then the angle AGH is greater than the angle GHD; to each of them add the angle BGH.

Therefore the angles BGH, AGH are greater than the angles BGH, GHD (Ax. 4).

But the angles BGH, AGH are together equal to two right angles (I. 3).

Therefore the angles BGH, GHD are less than two right angles. ∴ ∠ BGH + ∠ GHD < two right angles.

But if a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being

continually produced, shall at length meet on that side on which are the angles which are less than two right angles (Ax. 12);

Hence AB and CD meet, and are parallel.

Therefore the straight lines AB, CD will meet if produced far enough.

But they cannot meet, because they are parallel straight lines (Hyp.);

∴ ∠AGH not unequal to ∠GHD.

Therefore the angle AGH is not unequal to the angle GHD—that is, it is equal to it.

But the angle AGH is equal to the angle EGB (I. 15);

and ∠EGB = ∠GHD,

Therefore the angle EGB is equal to the angle GHD (Ax. 1).

Add to each of these the angle BGH.

Therefore the angles EGB, BGH, are equal to the angles BGH, GHD (Ax. 2).

But the angles EGB, BGH, are equal to two right angles (I. 13).

also ∠BGH + ∠GHD = two right angles.

Therefore also BGH, GHD, are equal to two right angles (Ax. 1).

Therefore, if a straight line, &c. *Q. E. D.*

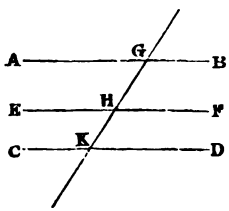
Proposition 30.—Theorem.

Straight lines which are parallel to the same straight lines are parallel to one another.

Let AB, CD be each of them parallel to EF;

AB shall be parallel to CD.

CONSTRUCTION.—Let the straight line GHK cut AB, EF, CD.



∠AGH or ∠AGK = ∠GHF,

PROOF.—Because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the angle GHF (I. 29).

and ∠GHF = ∠GKD.

Again, because GK cuts the parallel straight lines EF, CD, the angle GHF is equal to the angle GKD (I. 29).

And it was shown that the angle AGK is equal to the angle GHF;

∴ ∠AGK = ∠GKD.

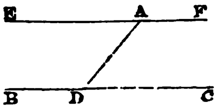
Therefore the angle AGK is equal to the angle GKD (Ax. 1), and they are alternate angles;

Therefore AB is parallel to CD (I. 27).
Therefore, straight lines, &c. *Q. E. D.*

Proposition 31.—Problem.

To draw a straight line through a given point, parallel to a given straight line.

Let A be the given point, and BC the given straight line.
It is required to draw a straight line through the point A, parallel to BC.



CONSTRUCTION.—In BC take any point D, and join AD.

At the point A, in the straight line AD, make the angle DAE equal to the angle ADC (I. 23). Make ∠ DAE = ∠ ADC.

Produce the straight line EA to F.

Then EF *shall be parallel to* BC.

PROOF.—Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another; They are alternate angles.

Therefore EF is parallel to BC (I. 27).

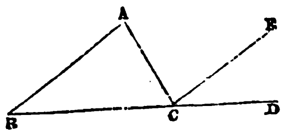
Therefore, the straight line EAF is drawn through the given point A, parallel to the given straight line BC. *Q. E. F.*

Proposition 32.—Theorem.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D;

The exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC;



And the three interior angles of the triangle—namely, ABC, BCA, CAB, shall be equal to two right angles.

CONSTRUCTION.—Through the point C, draw CE parallel to AB (I. 31). Make CE parallel to AB.

Then $\angle$ BAC = $\angle$ ACE,

Proof.—Because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal (I. 29).

and $\angle$ ECD = $\angle$ ABC.

Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and opposite angle ABC (I. 29).

But the angle ACE was shown to be equal to the angle BAC;

$\therefore$ $\angle$ ACD = $\angle$ BAC + $\angle$ ABC.

Therefore the whole exterior angle ACD is equal to the two interior and opposite angles BAC, ABC (Ax. 2).

Add $\angle$ ACB.

To each of these equals add the angle ACB.

Therefore the angles ACD, ACB are equal to the three angles CBA, BAC, ACB (Ax. 2).

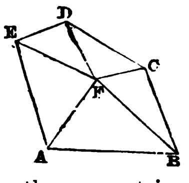
But the angles ACD, ACB are equal to two right angles (I. 13);

Therefore also the angles CBA, BAC, ACB are equal to two right angles (Ax. 1).

Therefore, if a side of any triangle, &c. *Q. E. D.*

Corollary 1.—*All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

For any rectilineal figure ABCDE can, by drawing straight lines from a point F within the figure to each angle, be divided into as many triangles as the figure has sides.



And, by the preceding proposition, the angles of each triangle are equal to two right angles.

Therefore all the angles of the triangles are equal to twice as many right angles as there are triangles; that is, as there are sides of the figure.

But the same angles are equal to the angles of the figure, together with the angles at the point F;

And the angles at the point F, which is the common vertex of all the triangles, are equal to four right angles (I. 15, Cor. 2);

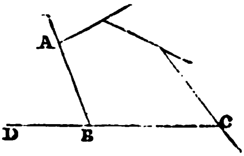
Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

COROLLARY 2.—*All the exterior angles of any rectilineal figure are together equal to four right angles.*

The interior angle ABC, with its adjacent exterior angle ABD, is equal to two right angles (I. 13);

Therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides.

But all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides (I. 32, Cor. 1);



Therefore all the interior angles, together with all the exterior angles, are equal to all the interior angles and four right angles (Ax. 1).

Take away the interior angles which are common;

Therefore all the exterior angles are equal to four right angles (Ax. 3).

Proposition 33.—Theorem.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.

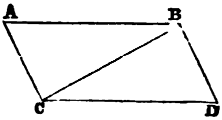
Let AB and CD be equal and parallel straight lines joined towards the same parts by the straight lines AC and BD;

AC and BD shall be equal and parallel.

CONSTRUCTION.—Join BC.

PROOF.—Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (I. 29). $\angle$ ABC = $\angle$ BCD.

Because AB is equal to CD, and BC common to the two triangles ABC, DCB, the two sides AB, BC are equal to the two sides DC, CB, each to each;



And the angle ABC was proved to be equal to the angle BCD;

Therefore the base AC is equal to the base BD (I. 4), $\therefore$ AC = BD,

And the triangle ABC is equal to the triangle BCD (I. 4),

And the other angles are equal to the other angles, each to each, to which the equal sides are opposite;

and ∠ ACB = ∠ CBD. Therefore the angle ACB is equal to the angle CBD.

And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another;

Therefore AC is parallel to BD (I. 27); and it was shown to be equal to it.

Therefore, the straight lines, &c. *Q. E. D.*

Proposition 34.—Theorem.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram—that is, divides it into two equal parts.

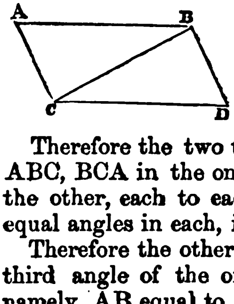
Let ACDB be a parallelogram, of which BC is a diagonal;

The opposite sides and angles of the figure shall be equal to one another,

And the diagonal BC shall bisect it.

PROOF.—Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal to one another (I. 29);

∠ ABC = ∠ BCD,



Because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal to one another (I. 29);

and ∠ ACB = ∠ CBD.

Therefore the two triangles ABC, BCD have two angles, ABC, BCA in the one, equal to two angles, BCD, CBD in the other, each to each; and the side BC, adjacent to the equal angles in each, is common to both triangles.

Therefore the other sides are equal, each to each, and the third angle of the one to the third angle of the other—namely, AB equal to CD, AC to BD, and the angle BAC to the angle CDB (I. 26).

∴ AB = CD, AC = BD, ∠ BAC = ∠ CDB,

And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB,

Therefore the whole angle ABD is equal to the whole angle ACD (Ax. 2).

and ∴ ∠ ABD = ∠ ACD,

And the angle BAC has been shown to be equal to the angle BDC; therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diagonal bisects it.

For AB being equal to CD, and BC common,

The two sides AB, BC are equal to the two sides CD and CB, each to each.

And the angle ABC has been shown to be equal to the angle BCD;

Therefore the triangle ABC is equal to the triangle BCD (I. 4), also △ ABC = △ BCD.

And the diagonal BC divides the parallelogram ABCD into two equal parts.

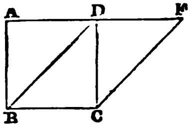
Therefore, the opposite sides, &c. *Q. E. D.*

Proposition 35.—Theorem.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

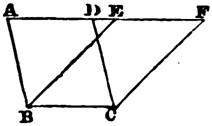
Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels AF, BC;

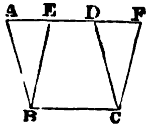
The parallelogram ABCD shall be equal to the parallelogram EBCF.



CASE 1.—If the sides AD, DF of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D, it is plain that each of the parallelograms is double of the triangle DBC (I. 34), and that they are therefore equal to one another (Ax. 6).

CASE 2.—But if the sides AD, EF, opposite to the base BC, of the parallelograms ABCD, EBCF, be not terminated in the same point, then—





PROOF.—Because ABCD is a parallelogram, AD is equal to BC (I. 34). AD = BC.

For the same reason EF is equal to BC; EF = BC.

Therefore AD is equal to EF (Ax. 1), and DE is common;

Therefore the whole, or the remainder, AE, is equal to the whole, or the remainder, DF (Ax. 2, or 3), ∴AE=DF.

And AB is equal to DC (I. 34).

Therefore the two EA, AB are equal to the two FD, DC, each to each;

And the exterior angle FDC is equal to the interior EAB (I. 29);

Therefore the base EB is equal to the base FC (I. 4),

Hence △ EAB = △ FDC.

And the triangle EAB equal to the triangle FDC (I. 4).

Take the triangle FDC from the trapezium ABCF, and from the same trapezium ABCF, take the triangle EAB, and the remainders are equal (Ax. 3)

That is, the parallelogram ABCD is equal to the parallelogram EBCF:

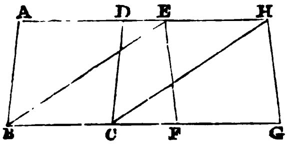
Therefore, parallelograms, &c. *Q. E. D.*

Proposition 36.—Theorem.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG;

The parallelogram ABCD shall be equal to the parallelogram EFGH.



CONSTRUCTION.—Join BE, CH.

PROOF.—Because BC is equal to FG (Hyp.), and FG to EH (I. 34),

BC = EH,

Therefore BC is equal to EH (Ax. 1); and they are parallels, and joined towards the same parts by the straight lines BE, CH.

But straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel (I. 33);

and BE = CH.

Therefore BE, CH are both equal and parallel;

EBCH a parallelogram, equal each of the given ones.

Therefore EBCH is a parallelogram (Def. 35),

And it is equal to the parallelogram ABCD, because they are on the same base BC, and between the same parallels BC, AH (I. 35).

For the like reason, the parallelogram EFGH is equal to the same parallelogram EBCH;

Therefore the parallelogram ABCD is equal to the parallelogram EFGH (Ax. 1).

Therefore, parallelograms, &c. *Q. E. D.*

Proposition 37.—Theorem.

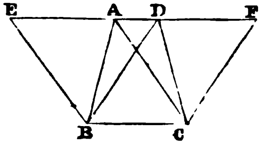
Triangles upon the same base, and between the same parallels, are equal to one another.

Let the triangles ABC, DBC be on the same base BC, and between the same parallels AD, BC;

The triangle ABC shall be equal to the triangle DBC.

Construction.—Produce AD both ways, to the points E, F.

Through B draw BE parallel to CA, and through C draw CF parallel to BD (I. 31).



Proof. — Then each of the figures EBCA, DBCF, is a parallelogram (Def. 35), and they are equal to one another, because they are on the same base BC, and between the same parallels BC, EF (I. 35.); — Figures EBCA and DBCF are equal;

And the triangle ABC is half of the parallelogram EBCA, because the diagonal AB bisects it (I. 34); — and the triangles are respectively half of these.

And the triangle DBC is half of the parallelogram DBCF, because the diagonal DC bisects it (I. 34).

But the halves of equal things are equal (Ax. 7);

Therefore the triangle ABC is equal to the triangle DBC.

Therefore, triangles, &c. *Q. E. D.*

Proposition 38.—Theorem.

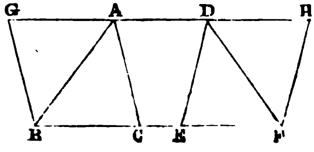
Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF, be on equal bases BC, EF, and between the same parallels BF, AD.

The triangle ABC shall be equal to the triangle DEF.

Construction.—Produce AD both ways to the points G, H.

Through B draw BG parallel to CA, and through F draw FH parallel to ED (I. 31).



Proof.—Then each of the figures GBCA, DEFH, is a — Figures GBCA and DEFH are equal;

parallelogram (Def. 35), and they are equal to one another, because they are on equal bases BC, EF, and between the same parallels BF, GH (I. 36);

and the triangles are half of these respectively.

And the triangle ABC is half of the parallelogram GBCA, because the diagonal AB bisects it (I. 34);

And the triangle DEF is half of the parallelogram DEFH, because the diagonal DF bisects it (I. 34).

But the halves of equal things are equal (Ax. 7);

Therefore the triangle ABC is equal to the triangle DEF.

Therefore, triangles, &c. *Q. E. D.*

Proposition 39.—Theorem.

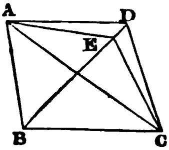
Equal triangles upon the same base, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be upon the same base BC, and on the same side of it;

They shall be between the same parallels.

CONSTRUCTION.—Join AD; AD shall be parallel to BC.

AE parallel to BC suppose.



For if it is not, through A draw AE parallel to BC (I. 31), and join EC.

PROOF.—The triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37).

Then △ DBC = △ EBC, an absurdity.

But the triangle ABC is equal to the triangle DBC (Hyp.);

Therefore the triangle DBC is equal to the triangle EBC (Ax. 1), the greater equal to the less, which is impossible;

Therefore AE is not parallel to BC.

In the same manner, it can be demonstrated that no line passing through A can be parallel to BC, except AD;

Therefore AD is parallel to BC.

Therefore, equal triangles, &c. *Q. E. D.*

Proposition 40.—Theorem.

Equal triangles upon the same side of equal bases, that are in the same straight line, are between the same parallels.

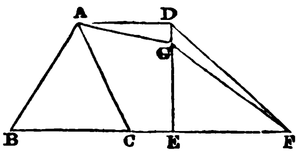
Let the equal triangles ABC, DEF, be upon the same side of equal bases BC, EF, in the same straight line BF.

The triangles ABC, DEF shall be between the same parallels.

CONSTRUCTION.—Join AD; AD shall be parallel to BF.

For if it is not, through A draw AG parallel to BF (I. 31), and join GF. AG parallel to BF suppose.

PROOF.—The triangle ABC is equal to the triangle GEF, because they are upon equal bases BC, EF, and are between the same parallels BF, AG (I. 38).



But the triangle ABC is equal to the triangle DEF;

Therefore the triangle DEF is equal to the triangle GEF (Ax. 1), the greater equal to the less, which is impossible; △ DEF = △ GEF, an absurdity.

Therefore AG is not parallel to BF.

In the same manner, it can be demonstrated that no line, passing through A, can be parallel to BF, except AD;

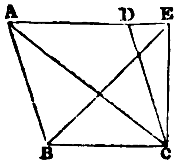
Therefore AD is parallel to BF.

Therefore, equal triangles, &c.

Proposition 41.—Theorem.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE;



The parallelogram ABCD shall be double of the triangle EBC.

CONSTRUCTION.—Join AC.

PROOF.—The triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37). △ ABC = △ EBC.

But the parallelogram ABCD is double of the triangle ABC, because the diagonal AC bisects the parallelogram (I. 34). And parallelogram = 2 △ ABC.

Therefore the parallelogram ABCD is also double of the triangle EBC (Ax. 1).

Therefore, if a parallelogram, &c. *Q. E. D.*

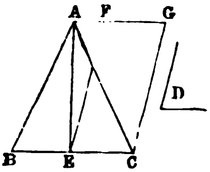
Proposition 42.—Problem.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle;

It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.

Make BE = EC and ∠ CEF = D



CONSTRUCTION.—Bisect BC in E (I. 10), and join AE.

At the point E, in the straight line CE, make the angle CEF equal to D (I. 23).

Through A draw AFG parallel to EC (I. 31).

Through C draw CG parallel to EF (I. 31).

Then FECG *is the parallelogram required.*

PROOF.—Because BE is equal to EC (Const.), the triangle ABE is equal to the triangle AEC, since they are upon equal bases and between the same parallels (I. 38);

△ ABC = 2 △ AEC, and also figure FECG = 2 △ AEC.

Therefore the triangle ABC is double of the triangle AEC.

But the parallelogram FECG is also double of the triangle AEC, because they are upon the same base, and between the same parallels (I. 41);

Therefore the parallelogram FECG is equal to the triangle ABC (Ax. 6),

And it has one of its angles CEF equal to the given angle D (Const.).

Therefore a parallelogram FECG has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D. *Q. E. F.*

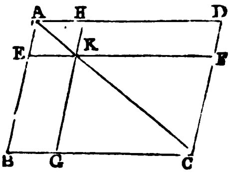
Proposition 43.—Theorem.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to one another.

Let ABCD be a parallelogram, of which the diagonal is AC; and EH, GF parallelograms about AC, that is, through which AC passes; and BK, KD the other parallelograms, which make up the whole figure ABCD, and are therefore called the complements.

The complement BK shall be equal to the complement KD.

PROOF.—Because ABCD is a parallelogram, and AC its diagonal, the triangle ABC is equal to the triangle ADC (I. 34). △ ABC = △ ADC.



Again, because AEKH is a parallelogram, and AK its diagonal, the triangle AEK is equal to the triangle AHK (I. 34). Also △ AEK = △ AHK.

For the like reason the triangle KGC is equal to the triangle KFC. And △ KGC = △ KFC.

Therefore, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC;

The triangles AEK, KGC are equal to the triangles AHK, KFC (Ax. 2).

But the whole triangle ABC was proved equal to the whole triangle ADC;

Therefore the remaining complement BK is equal to the remaining complement KD (Ax. 3). ∴ BK = KD.

Therefore, the complements, &c. *Q. E. D.*

Proposition 44.—Problem.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, C the given triangle, and D the given angle.

It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

Make parallelogram BEFG = ∠ C, and ∠ at B = ∠ D, and EBA a straight line.

CONSTRUCTION 1.—Make the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D (I. 42);

And let the parallelogram BEFG be made so that BE may be in the same straight line with AB.

Produce FG to H.

Through A draw AH parallel to BG or EF (I. 31).

Join HB.

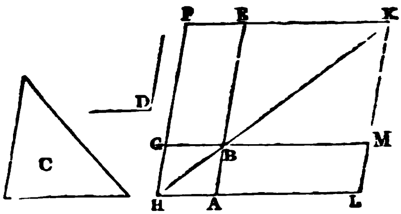
PROOF 1.—Because the straight line HF falls on the parallels AH, EF, the angles AHF, HFE are together equal to two right angles (I. 29).

Therefore the angles BHF, HFE are together less than two right angles (Ax. 9). But straight lines which with another straight line make the interior angles on the same side together less than two right angles, will meet on that side, if produced far enough (Ax. 12);

HB and FE meet.

Therefore HB and FE shall meet if produced.

CONSTRUCTION 2.—Produce HB and FE towards BE, and let them meet in K.



Through K draw KL parallel to EA or FH (I. 31).

Produce HA, GB to the points L, M.

Then LB *shall be the parallelogram required.*

PROOF 2.—Because HLKF is a parallelogram, of which the diagonal is HK; and AG, ME are the parallelograms about HK; and LB, BF are the complements;

Figures LB = BF.

Therefore the complement LB is equal to the complement BF (I. 43).

But BF is equal to the triangle C (Const.);
Therefore LB is equal to the triangle C (Ax. 1).

But BF = $\triangle$ C, $\therefore$ LB = $\triangle$ C;

And because the angle GBE is equal to the angle ABM (I. 15), and likewise to the angle D (Const.);

Therefore the angle ABM is equal to the angle D (Ax. 1).

Also $\angle$ ABM = $\angle$ GBE = $\angle$ D.

Therefore, the parallelogram LB is applied to the straight line AB, and is equal to the triangle C, and has the angle ABM equal to the angle D. *Q. E. F.*

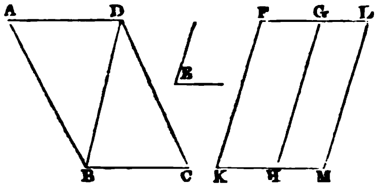
Proposition 45.—Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle.

It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

Construction.—Join DB.



Describe the parallelogram FH equal to the triangle ADB, and having the angle FKH equal to the angle E (I. 42).

Make FH = $\triangle$ ADB. Apply to GH, GM = $\triangle$ DBC, with $\angle$ GHM = $\angle$ E.

To the straight line GH apply the parallelogram GM equal to the triangle DBC, and having the angle GHM equal to the angle E (I. 44).

Then the figure FKML *shall be the parallelogram required.*

Proof.—Because the angle E is equal to each of the angles FKH, GHM (Const.),

Therefore the angle FKH is equal to the angle GHM (Ax. 1).

Add to each of these equals the angle KHG;

Therefore the angles FKH, KHG are equal to the angles KHG, GHM (Ax. 2).

But FKH, KHG are equal to two right angles (I. 29);

Therefore also KHG, GHM are equal to two right angles (Ax. 1).

And because at the point H, in the straight line GH, the two straight lines KH, HM, on the opposite sides of it, make the adjacent angles together equal to two right angles,

Then KHM is a straight line;

Therefore KH is in the same straight line with HM (I. 14).

And because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal (I. 29).

Add to each of these equals the angle HGL;

Therefore the angles MHG, HGL are equal to the angles HGF, HGL (Ax. 2).

But the angles MHG, HGL are equal to two right angles (I. 29);

Therefore also the angles HGF, HGL are equal to two right angles,

and FGL is a straight line.

And therefore FG is in the same straight line with GL (I. 14).

And because KF is parallel to HG, and HG parallel to ML (Const.);

Therefore KF is parallel to ML (I. 30).

And KM, FL are parallels (Const);

∴ KFLM a parallelogram.

Therefore KFLM is a parallelogram (Def. 35).

And because the triangle ABD is equal to the parallelogram HF, and the triangle DBC equal to the parallelogram GM (Const.),

And the figure ABCD is equal to it.

Therefore the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM (Ax. 2).

Therefore, the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E. *Q. E. F.*

Corollary.—From this it is manifest how to apply to a given straight line a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure—namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle; and so on (I. 44).

Proposition 46.—Problem.

To describe a square upon a given straight line.

Let AB be the given straight line;

It is required to describe a square upon AB.

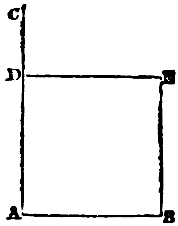
CONSTRUCTION.—From the point A draw AC at right angles to AB (I. 11),

And make AD equal to AB (I. 3).

Through the point D draw DE parallel to AB (I. 31).

Through the point B draw BE parallel to AD (I. 31).

Then ADEB *shall be the square required.*



PROOF.—Because DE is parallel to AB, and BE parallel to AD (Const.), therefore ADEB is a parallelogram; ADEB a parallelogram.

Therefore AB is equal to DE, and AD to BE (I. 34).

But AB is equal to AD (Const.);

Therefore the four straight lines BA, AD, DE, EB are equal to one another (Ax. 1),

And the parallelogram ADEB is therefore equilateral. It is equilateral.

Likewise all its angles are right angles.

For since the straight line AD meets the parallels AB, DE, the angles BAD, ADE are together equal to two right angles (I. 29).

But BAD is a right angle (Const.);

Therefore also ADE is a right angle (Ax. 3).

But the opposite angles of parallelograms are equal (I. 34);

Therefore each of the opposite angles ABE, BED is a right angle (Ax. 1);

Therefore the figure ADEB is rectangular; and it has been proved to be equilateral; therefore it is a square (Def. 30). It is rectangular.

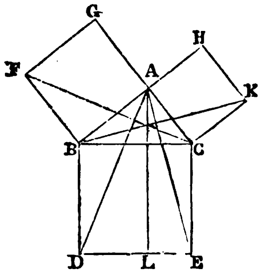
Therefore, the figure ADEB is a square, and it is described upon the given straight line AB. *Q. E. F.* ∴ a square.

COROLLARY.—Hence every parallelogram that has one right angle has all its angles right angles.

Proposition 47.—Theorem.

In any right-angled triangle, the square which is described upon the side opposite to the right angle is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC;



The square described upon the side BC shall be equal to the squares described upon BA, AC.

CONSTRUCTION. — On BC describe the square BDEC (I. 46).

On BA, AC describe the squares GB, HC (I. 46).

Through A draw AL parallel to BD or CE (I. 31).

Join AD, FC.

PROOF.—Because the angle BAC is a right angle (Hyp.), and that the angle BAG is also a right angle (Def. 30),

The two straight lines AC, AG, upon opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles;

C G is a straight line.

Therefore CA is in the same straight line with AG (I. 14).

BAH is a straight line.

For the same reason, AB and AH are in the same straight line.

Now the angle DBC is equal to the angle FBA, for each of them is a right angle (Ax. 11); add to each the angle ABC.

Therefore the whole angle DBA is equal to the whole angle FBC (Ax. 2).

And because the two sides AB, BD are equal to the two sides FB, BC, each to each (Def. 30), and the angle DBA equal to the angle FBC;

△ ABD = △ FBC.

Therefore the base AD is equal to the base FC, and the triangle ABD to the triangle FBC (I. 4).

Now the parallelogram BL is double of the triangle ABD, because they are on the same base BD, and between the same parallels BD, AL (I. 41).

And the square GB is double of the triangle FBC, because they are on the same base FB, and between the same parallels FB, GC (I. 41).

Hence parallelogram BL = square GB, and parallelogram CL = square HC.

But the doubles of equals are equal (Ax. 6), therefore the parallelogram BL is equal to the square GB.

In the same manner, by joining AE, BK, it can be shown that the parallelogram CL is equal to the square HC.

Therefore the whole square BDEC is equal to the two squares GB, HC (Ax. 2);

And the square BDEC is described on the straight line BC, and the squares GB, HC upon BA, AC. $\therefore BC^2 = BA^2 + AC^2$

Therefore the square described upon the side BC is equal to the squares described upon the sides BA, AC.

Therefore, in any right-angled triangle, &c. *Q. E. D.*

Proposition 48.—Theorem.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares described upon the other sides BA, AC;

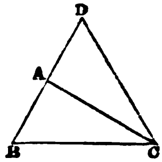
The angle BAC shall be a right angle.

Construction.—From the point A draw AD at right angles to AC (I. 11). Draw AD at right angles to AC.

Make AD equal to BA (I. 3), and join DC. *(Do not produce BA.)*

Proof.—Because DA is equal to AB, the square on DA is equal to the square on BA.

To each of these add the square on AC.



Therefore the squares on DA, AC are equal to the squares on BA, AC (Ax. 2).

But because the angle DAC is a right angle (Const.), the square on DC is equal to the squares on DA, AC (I. 47),

And the square on BC is equal to the squares on BA, AC (Hyp.);

Therefore the square on DC is equal to the square on BC (Ax. 1); Then $DC^2 = BC^2$, and $DC = BC$.

And therefore the side DC is equal to the side BC.

And because the side DA is equal to AB (Const.), and AC common to the two triangles DAC, BAC, the two sides DA, AC are equal to the two sides BA, AC, each to each.

And the base DC has been proved equal to the base BC;

Hence $\angle DAC = \angle BAC$.

Therefore the angle DAC is equal to BAC (I. 8).
But DAC is a right angle (Const.);
Therefore also BAC is a right angle (Ax. 1).
Therefore, if the square, &c. *Q. E. D.*

EXERCISES ON BOOK I.

Prop. 1—15.

1. From the greater of two given straight lines to cut off a portion which is three times as long as the less.

2. The line bisecting the vertical angle of an isosceles triangle also bisects the base.

3. Prove Euc. I. 5, by the method of *super-position.*

4. In the figure to Euc. I. 5, show that the line joining A with the point of intersection of BG and FC, makes equal angles with AB and AC.

5. ABC is an isosceles triangle, whose base is BC, and AD is perpendicular to BC; every point in AD is equally distant from B and C.

6. Show that the sum of the sum and difference of two given straight lines is twice the greater, and that the difference of the sum and difference is twice the less.

7. Prove the same property with regard to angles.

8. Make an angle which shall be three-fourths of a right angle.

9. If, with the extremities of a given line as centres, circles be drawn intersecting in two points, the line joining the points of intersection will be perpendicular to the given line, and will also bisect it.

10. Find a point which is at a given distance from a given point and from a given line.

11. Show that the sum of the angles round a given point are together equal to four right angles.

12. If the exterior angle of a triangle and its adjacent interior angle be bisected, the bisecting lines will be at right angles.

13. If three points, A, B, C, be taken not in the same straight line, and AB and AC be joined and bisected by perpendiculars which meet in D, show that DA, DB, DC are equal to each other.

Prop. 16—32.

14. The perpendiculars from the angular points upon the opposite sides of a triangle meet in a point.

15. To construct an isosceles triangle on a given base, the sides being each of them double the given base.

16. Describe an isosceles triangle having a given base, and whose vertical angle is half a right angle.

17. AB is a straight line, C and D are points on the same side of it; find a point E in AB such that the sum of CE and ED shall be a minimum.

18. Having given two sides of a triangle and an angle, construct the triangle. Examine the cases when there will be (1.) one solution; (2.) two solutions; (3.) none.

19. Given an angle of a triangle and the sum and difference of the two sides including the angle, to construct the triangle.

20. Show that each of the angles of an equilateral triangle is two-thirds of a right angle, and hence show how to trisect a right angle.

21. If two angles of a triangle be bisected by lines drawn from the angular points to a given point within, then the line bisecting the third angle will pass through the same point.

22. The difference of any two sides of a triangle is less than the third side.

23. If the angles at the base of a right-angled isosceles triangle be bisected, the bisecting line includes an angle which is three halves of a right angle.

24. The sum of the lines drawn from any point within a polygon to the angular points is greater than half the sum of the sides of the polygon.

Prop. 33—48.

25. Show that the diagonals of a square bisect each other at right angles, and that the square described upon a semi-diagonal is half the given square.

26. Divide a given line into any number of equal parts, and hence show how to divide a line similarly to a given line.

27. If D and E be respectively the middle points of the sides BC and AC of the triangle ABC, and AD and BE be joined, and intersect in G, show that GD and GE are respectively one-third of AD and BE.

28. The lines drawn to the bisections of the sides of a triangle from the opposite angles meet in a point.

29. Describe a square which is five times a given square.

30. Show that a square, hexagon, and dodecagon will fill up the space round a point.

31. Divide a square into three equal areas, by lines drawn parallel to one of the diagonals.

32. Upon a given straight line construct a regular octagon.

33. Divide a given triangle into equal triangles by lines drawn from one of the angles.

34. If any two angles of a quadrilateral are together equal to two right angles, show that the sum of the other two is two right angles.

35. The area of a trapezium having two parallel sides is equal to half the rectangle contained by the perpendicular distance between the parallel sides of the trapezium, and the sum of the parallel sides.

36. The area of any trapezium is half the rectangle contained by one of the diagonals of the trapezium, and the sum of the perpendiculars let fall upon it from the opposite angles.

37. If the middle points of the sides of a triangle be joined, the lines form a triangle whose area is one-fourth that of the given triangle.

38. If the sides of a triangle be such that they are respectively the sum of two given lines, the difference of the same two lines, and twice the side of a square equal to the rectangle contained by these lines, the triangle shall be right-angled, having the right angle opposite to the first-named side.

39. If a point be taken within a triangle such that the lengths of the perpendiculars upon the sides are equal, show that the area of the rectangle contained by one of the perpendiculars and the perimeter of the triangle is double the area of the triangle.

40. In the last problem, if O be the given point, and OD, OE, OF the respective perpendiculars upon the sides BC, AC, and AB, show that the sum of the squares upon AD, OB, and DC exceeds the sum of the squares upon AF, BD, and CD by three times the square upon either of the perpendiculars.

41. Having given the lengths of the segments AF, BD, CE, in Problem 40, construct the triangle.

42. Draw a line, the square upon which shall be seven times the square upon a given line.

43. Draw a line, the square upon which shall be equal to the sum or difference of two given squares.

44. Reduce a given polygon to an equivalent triangle.

45. Divide a triangle into equal areas by drawing a line from a given point in a side.

46. Do the same with a given parallelogram.

47. If in the fig., Euc. I. 47, the square on the hypothenuse be on the other side, show how the other two squares may be made to cover exactly the square on the hypothenuse.

48. The area of a quadrilateral whose diagonals are at right angles is half the rectangle contained by the diagonals.

49. Bisect a given triangle by a straight line drawn from one of its angles.

50. Do the same with a given rectilineal figure ABCDEF.

51. If from the angle A of a triangle ABC a perpendicular be drawn meeting the base or base produced in D, show that the difference of the squares of AB and AC is equal to the difference of the squares of BD and DC.

52. If a straight line join the points of bisection of two sides of a triangle, the base is double the length of this line.

53. ABCD is a parallelogram, and E a point within it, and lines are drawn through E parallel to the sides of the parallelogram, show that E must lie on the diagonal AC when the figures BC and DE are equal.

54. If AD, BE, CF, are the perpendiculars from the angular points

of the triangle ABC upon the opposite sides, show that the sum of the squares upon AE, CD, BF, is equal to the sum of the squares upon CE, BD, AF.

55. The diagonals of a parallelogram bisect one another.

56. Write out at full length a definition of parallelism, and then prove that the alternate angles are equal when a straight line meets two parallel straight lines.

57. ABCDE are the angular points of a regular pentagon, taken in order. Join AC and BD meeting in H, and show that AEDH is an equilateral parallelogram.

58. Having given the middle points of the sides of a triangle, show how to construct the triangle.

59. Show that the diagonal of a parallelogram diminishes while the angle from which it is drawn increases. What is the limit to which the diagonal approaches as the angle approaches respectively zero and two right angles?

60. A, B, C, are three angles taken in order of a regular hexagon, show that the square on AC is three times the square upon a side of the hexagon.

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